

GLOBAL HIGHER INTEGRABILITY FOR PARABOLIC QUASIMINIMIZERS IN METRIC SPACES

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ABSTRACT. We prove higher integrability up to the boundary for minimal p -weak upper gradients of parabolic quasiminimizers in metric measure spaces, related to the heat equation. We assume the underlying metric measure space to be equipped with a doubling measure and to support a weak Poincaré-inequality.

1. INTRODUCTION

The problem of finding a solution to the classical heat equation

$$-\frac{\partial u}{\partial t} + \Delta u = 0,$$

in a parabolic cylinder $\Omega \times (0, T)$ can be reformulated into the variational problem of finding a function u such that with $K = 1$ we have

$$(1.1) \quad 2 \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t} dx dt + \int_{\{\phi \neq 0\}} |\nabla u|^2 dx dt \leq K \int_{\{\phi \neq 0\}} |\nabla(u + \phi)|^2 dx dt,$$

for all compactly supported test functions $\phi \in C_0^\infty(\Omega \times (0, T))$. Here Ω denotes a bounded domain in \mathbb{R}^d . A generalization of this minimization problem is to consider inequality (1.1) with the relaxed assumption $K \geq 1$: a function u satisfying this generalized condition is then called a parabolic quasiminimizer [Wie] related to the heat equation.

Our main result, Theorem 5.2, is to show that if u is a parabolic quasiminimizer in the general metric measure space setting, and satisfies a Dirichlet type parabolic boundary condition, where the domain is assumed to be regular enough, then u has the following global higher integrability property: The upper gradient [HeK] of u is integrable over the whole cylinder $\Omega \times (0, T)$ to a slightly higher power than initially assumed.

Assuming a weak Poincaré inequality, a doubling measure and a thickness condition for the complement of the domain Ω , we prove a parabolic Poincaré and Caccioppoli type estimate for u up to the boundary. Then we

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combine these with Sobolev's inequality and a self improving property for the thickness condition [L],[BMS] to establish a reverse Hölder inequality up to the boundary.

The novelty of this paper is that we prove these estimates in the general metric measure space setting, using a purely variational approach. No reference is made to the concept of a weak solution, or to the explicit scaling properties of the measure. Furthermore, no assumptions of translation invariance or absolute continuity of the underlying measure are made. Instead we base the proofs on taking integral averages and on the doubling property of the measure. On the other hand, the concept of parabolic quasiminimizers is extended into metric spaces by replacing gradients with the more general concept of upper gradients, which do not require the existence of partial derivatives.

The concept of quasiminimizers originates from the elliptic case in Euclidean spaces, where Giaquinta and Giusti showed in their celebrated papers [GG2, GG3] that many properties of weak solutions to elliptic PDEs generalize to a class of elliptic quasiminimizers

$$\int_{\{\phi \neq 0\}} |\nabla u|^2 \, dx \leq K \int_{\{\phi \neq 0\}} |\nabla(u + \phi)|^2 \, dx.$$

Hence, to some extent quasiminimizers provide a unifying approach to the theory of elliptic nonlinear partial differential equations. Indeed, for example a solution u to

$$\operatorname{div} F(\nabla u) = 0$$

under suitable regularity assumptions and the growth bounds

$$\alpha |\nabla u|^2 \leq F(\nabla u) \leq \beta |\nabla u|^2,$$

is a quasiminimizer with a constant $K = \beta/\alpha$. However, in other respects, the theory of quasiminimizers differs from that of minimizers. For example, quasiminimizers do not provide a unique solution to the Dirichlet problem, and they do not obey the comparison principle. One advantage of quasiminimizers is that they allow for replacing the gradients with a comparable concept which is definable in a more general setting. This way, by using upper gradients, Kinnunen and Shanmugalingam [KS] were able to extend the concept of quasiminimizers to metric measure spaces.

Following Giaquinta and Giusti, parabolic quasiminimizers were introduced in the Euclidean setting by Wieser [Wie]. In recent papers [KMMP], [MS], [MM], [MMPP], following Kinnunen and Shanmugalingam, the definition and study of parabolic quasiminimizers has been extended to metric measure spaces. In this paper we follow the same approach.

Substantial progress was made in the mid-1950s and -1960s in the regularity theory of elliptic equations due to the discoveries of De Giorgi [DG1], Nash [N] and Moser [Mos1, Mos2]. A natural question was, whether these results extend to systems as well. Morrey [Mor] proved that up to a set of measure zero a solution to a elliptic systems is regular. However, it was soon

discovered by De Giorgi [DG2] followed by Giusti and Miranda [GM], that full regularity for systems actually fails, and thus the partial regularity is best one can, in general, hope for.

The generalizations of Morrey's partial regularity result (Giaquinta and Giusti [GG1] as well as Giaquinta and Modica [GM]) rely on the higher integrability of the gradient. Such results for elliptic PDEs were obtained by Bojarski [Bo] as well as Meyers [Me], and by Gehring [G] in the context of quasiconformal mappings. In [EM], Elcrat and Meyers proved the local higher integrability for nonlinear elliptic systems. Later, in [GS82], Giaquinta and Struwe studied similar questions for systems of parabolic equations with quadratic growth conditions, and in [KL] Kinnunen and Lewis showed that p -parabolic type systems share the higher integrability property as well.

Another natural direction to extend regularity results is to consider regularity up to the boundary. Already elliptic examples in [KK] demonstrate that both the regularity of the boundary as well as the boundary values play a role in the proofs. Recently, local and global higher integrability questions have inspired an extensive literature, see for example [Gr], [Wie], [A], [Mi], [AM], [P1], [Bö], [Bö], [P3], [P2], [BP], [BR], [BRW], [BDM], [F], and [H].

2. PRELIMINARIES

2.1. Doubling measure. Let $X = (X, d, \mu)$ be a complete linearly locally convex metric space endowed with a positive doubling Borel measure μ which supports a weak $(1, 2)$ -Poincaré inequality.

The measure μ is called *doubling* if there exists a constant $c_\mu \geq 1$, such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$\mu(2B) \leq c_\mu \mu(B),$$

where $\lambda B = B(x_0, \lambda r)$. By iterating the doubling condition, it follows with $s = \log_2 c_\mu$ and $C = c_\mu^{-2}$ that

$$(2.1) \quad \frac{\mu(B(z, r))}{\mu(B(y, R))} \geq C \left(\frac{r}{R} \right)^s,$$

for all balls $B(y, R) \subset X$, $z \in B(y, R)$ and $0 < r \leq R < \infty$. However, the choice $s = \log_2 c_\mu$ may not be optimal, and we just assume that s is any number such that (2.1) is satisfied. From now on, throughout this text we assume that $c_\mu > 1$ and so $s > 0$.

A metric space X is called linearly locally convex if there exists constants $C_1 > 0$ and $r_1 > 0$ such that for all balls B in X with radius at most r_1 , every pair of distinct points in the annulus $2B \setminus \overline{B}$ can be connected by a curve lying in the annulus $2C_1 B \setminus C_1^{-1} \overline{B}$, see Section 3.12 in [HeK] and [BMS]. The assumption that X is linearly locally convex will be needed for Theorem 4.5 below.

2.2. Notation. Next we introduce more notation used throughout this paper. Given any $z_0 = (x_0, t_0) \in X \times \mathbb{R}$ and $\rho > 0$, let

$$B_\rho(x_0) = \{x \in X : d(x, x_0) < \rho\},$$

denote an open ball in X , and let

$$\Lambda_\rho(t_0) = (t_0 - \frac{1}{2}\rho^2, t_0 + \frac{1}{2}\rho^2),$$

denote an open interval in \mathbb{R} . A space-time cylinder in $X \times \mathbb{R}$ is denoted by

$$Q_\rho(z_0) = B_\rho(x_0) \times \Lambda_\rho(t_0),$$

so that $\nu(Q_\rho(z_0)) = \mu(B_\rho(x_0))\rho^2$. When no confusion arises, we shall omit the reference points and write briefly B_ρ , Λ_ρ and Q_ρ . We denote the product measure by $d\nu = d\mu dt$. The integral average of u is denoted by

$$(2.2) \quad u_{B_\rho}(t) = \int_{B_\rho} u(x, t) d\mu = \frac{1}{\mu(B_\rho)} \int_{B_\rho} u(x, t) d\mu$$

and

$$\int_{Q_\rho} u d\nu = \frac{1}{\nu(Q_\rho)} \int_{Q_\rho} u d\nu.$$

2.3. Upper gradients. Following [HeK], a non-negative Borel measurable function $g : X \rightarrow [0, \infty]$ is said to be an upper gradient of a function $u : X \rightarrow [-\infty, \infty]$, if for all compact rectifiable paths γ joining x and y in X we have

$$(2.3) \quad |u(x) - u(y)| \leq \int_\gamma g ds.$$

In case $u(x) = u(y) = \infty$ or $u(x) = u(y) = -\infty$, the left side is defined to be ∞ . Assume $1 \leq p < \infty$. The p -modulus of a family of paths Γ in X is defined to be

$$\inf_\rho \int_X \rho^p d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ , we have

$$\int_\gamma \rho ds \geq 1.$$

A property is said to hold for p -almost all paths, if the set of non-constant paths for which the property fails is of zero p -modulus. Following [KM, Sh1], if (2.3) holds for p -almost all paths γ in X , then g is said to be a p -weak upper gradient of u .

When $1 < p < \infty$ and $u \in L^p(X)$, it can be shown [Sh2] that there exists a minimal p -weak upper gradient of u , we denote it by g_u , in the sense that g_u is a p -weak upper gradient of u and for every p -weak upper gradient g of u it holds $g_u \leq g$ μ -almost everywhere in X . Moreover, if $v = u$ μ -almost

everywhere in a Borel set $A \subset X$, then $g_v = g_u$ μ -almost everywhere in X . Also, if $u, v \in L^p(X)$, then μ -almost everywhere in X , we have

$$\begin{aligned} g_{u+v} &\leq g_u + g_v, \\ g_{uv} &\leq |u|g_v + |v|g_u. \end{aligned}$$

Proofs for these properties and more on upper gradients in metric spaces can be found for example in [BB] and the references therein. See also [C] for a discussion on upper gradients.

2.4. Newtonian spaces. Following [Sh1], for $1 < p < \infty$, and $u \in L^p(\Omega)$ where $\Omega \subset X$ is a domain, we define

$$\|u\|_{1,p,\Omega}^p = \|u\|_{L^p(\Omega,\mu)}^p + \|g_u\|_{L^p(\Omega,\mu)}^p,$$

and

$$\tilde{N}^{1,p}(\Omega) = \{u : \|u\|_{1,p,\Omega} < \infty\}.$$

An equivalence relation in $\tilde{N}^{1,p}(\Omega)$ is defined by saying that $u \sim v$ if

$$\|u - v\|_{\tilde{N}^{1,p}(\Omega)} = 0.$$

The *Newtonian space* $N^{1,p}(\Omega)$ is defined to be the space $\tilde{N}^{1,p}(\Omega)/\sim$, with the norm

$$\|u\|_{N^{1,p}(\Omega)} = \|u\|_{1,p,\Omega}.$$

A function u belongs to the local Newtonian space $N_{\text{loc}}^{1,p}(\Omega)$ if it belongs to $N^{1,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$. The Newtonian space with zero boundary values is defined as $N_0^{1,p}(\Omega) = \{f \in N^{1,p}(\Omega) : f \text{ can be continued into a function in } N^{1,p}(X) \text{ by setting } f = 0 \text{ outside } \Omega\}$. For more properties of Newtonian spaces, see [He, Sh1, BB].

2.5. Poincaré's and Sobolev's inequality. For $1 \leq q < \infty$, $1 < p < \infty$, the measure μ is said to support a weak (q, p) -Poincaré inequality if there exist constants $c_P > 0$ and $\lambda \geq 1$ such that

$$(2.4) \quad \left(\int_{B_\rho(x)} |v - v_{B_\rho(x)}|^q d\mu \right)^{1/q} \leq c_P \rho \left(\int_{B_{\lambda\rho}(x)} g_v^p d\mu \right)^{1/p},$$

for every $v \in N^{1,p}(X)$ and $B_\rho(x) \subset X$. In case $\lambda = 1$, we say a (q, p) -Poincaré inequality is in force. In a general metric measure space setting, it is of interest to have assumptions which are invariant under bi-Lipschitz mappings. The weak (q, p) -Poincaré inequality has this quality.

For a metric space X equipped with a doubling measure μ , it is a result by Hajlasz and Koskela [HaK1] that the following Sobolev inequality holds:

If X supports a weak $(1, p)$ -Poincaré inequality for some $1 < p < \infty$, then X also supports a weak (κ, p) -Poincaré inequality, where

$$\kappa = \begin{cases} \frac{d_\mu p}{d_\mu - p}, & \text{for } 1 < p < d_\mu, \\ 2p, & \text{otherwise,} \end{cases}$$

possibly with different constants $c'_p > 0$ and $\lambda' \geq 1$.

Remark 2.1. It is a recent result by Keith and Zhong [KZ], that when $1 < p < \infty$ and (X, d) is a complete metric space with doubling measure μ , the weak $(1, p)$ -Poincaré inequality implies a weak $(1, q)$ -Poincaré inequality for some $1 < q < p$. Then by the above discussion, X also supports a weak (κ, q) -Poincaré inequality with $\kappa > q$ as above. By Hölder's inequality, we can assume that q is close enough to p , so that $\kappa \geq p$. By Hölder's inequality, the left hand side of the weak (κ, q) -Poincaré inequality can be estimated from below by replacing κ with any positive $\kappa' < \kappa$. Hence we conclude, that if X supports a weak $(1, p)$ -Poincaré inequality with $1 < p < \infty$, then X also supports a weak (p, q) -Poincaré and a weak (q, q) -Poincaré inequality with some $1 < q < p$.

2.6. Parabolic spaces and upper gradients. For $1 < p < \infty$, we say that

$$u \in L^p(0, T; N^{1,p}(\Omega)),$$

if the function $x \mapsto u(x, t)$ belongs to $N^{1,p}(\Omega)$ for almost every $0 < t < T$, and $u(x, t)$ is measurable as a mapping from $(0, T)$ to $N^{1,p}(\Omega)$, that is, the preimage on $(0, T)$ for any given open set in $N^{1,p}(\Omega)$ is measurable. Furthermore, we require that the norm

$$\|u\|_{L^p(0,T;N^{1,p}(\Omega))} = \left(\int_0^T \|u\|_{N^{1,p}(\Omega)}^p dt \right)^{1/p}$$

is finite. Analogously, we define $L^p(0, T; N_0^{1,p}(\Omega))$ and $L_{\text{loc}}^p(0, T; N_{\text{loc}}^{1,p}(\Omega))$. The space of compactly supported Lipschitz-continuous functions $\text{Lip}_c(\Omega_T)$ consists of functions u , $\text{supp } u \subset \Omega_T$, for which there exists a positive constant $C_{\text{Lip}}(u)$ such that

$$|u(x, t) - u(y, s)| \leq C_{\text{Lip}}(u)(d(x, y) + |t - s|),$$

whenever $(x, t), (y, s) \in \Omega_T$. The parabolic minimal p -weak upper gradient of a function $u \in L_{\text{loc}}^p(t_1, t_2; N_{\text{loc}}^{1,p}(\Omega))$ is defined in a natural way by setting

$$g_u(x, t) = g_{u(\cdot, t)}(x),$$

at ν -almost every $(x, t) \in \Omega \times (0, T)$. When u depends on time, we refer to g_u as the upper gradient of u . The next Lemma on taking limits of upper gradients will be used later in this paper. Here and throughout this paper we denote the time wise mollification of a function by

$$f_\varepsilon(x, t) = \int_{-\varepsilon}^{\varepsilon} \zeta_\varepsilon(s) f(x, t - s) ds,$$

where ζ_ε is the standard mollifier with support in $(-\varepsilon, \varepsilon)$.

Lemma 2.2. *Let $u \in L_{loc}^p(0, T; N_{loc}^{1,p}(\Omega))$, where $1 < p < \infty$. Then the following statements hold:*

- (a) *As $s \rightarrow 0$, we have $g_{u(x,t-s)-u(x,t)} \rightarrow 0$ in $L_{loc}^p(\Omega_T)$.*
- (b) *As $\varepsilon \rightarrow 0$, we have $g_{u_\varepsilon-u} \rightarrow 0$ pointwise ν -almost everywhere in Ω_T and in $L_{loc}^p(\Omega_T)$.*

Proof. See Lemma 6.8 in [MS]. □

2.7. Parabolic quasiminimizers.

Definition 2.3. Let Ω be an open subset of X , $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ and $K' \geq 1$. A function u belonging to the parabolic space $L_{loc}^2(0, T; N_{loc}^{1,2}(\Omega))$ is a *parabolic quasiminimizer* if

$$\int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t} d\nu + \int_{\{\phi \neq 0\}} E(u) d\nu \leq K' \int_{\{\phi \neq 0\}} E(u + \phi) d\nu,$$

for every $\phi \in \text{Lip}(\Omega_T)$ such that $\{\phi \neq 0\} \subset\subset \Omega_T$, where we denote $E(u) = F(x, t, g)$ and $F : \Omega \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following assumptions:

- (1) $(x, t) \mapsto F(x, t, \xi)$ is measurable for every ξ ,
- (2) $\xi \mapsto F(x, t, \xi)$ is continuous for almost every (x, t) ,
- (3) there exist $0 < c_1 \leq c_2 < \infty$ such that for every ξ and almost every (x, t) , we have

$$c_1 |\xi|^2 \leq F(x, t, \xi) \leq c_2 |\xi|^2.$$

As a consequence of the above, a parabolic quasiminimizer u satisfies

$$(2.5) \quad \alpha \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t} d\nu + \int_{\{\phi \neq 0\}} g_u^2 d\nu \leq K \int_{\{\phi \neq 0\}} g_{u+\phi}^2 d\nu,$$

with $K = c_2 c_1^{-1} K' \geq 1$ and $\alpha = c_1^{-1}$, for every $\phi \in \text{Lip}(\Omega_T)$ such that $\{\phi \neq 0\} \subset\subset \Omega$. There is a subtle difficulty in proving Caccioppoli type estimates in the parabolic case: one often needs a test function depending on u itself, but u is a priori not necessarily in $\text{Lip}(\Omega_T)$ nor has compact support. We treat this difficulty in the following manner. Consider a test function $\phi \in \text{Lip}(\Omega_T)$ with compact support. By a change of variable in (2.5), we see that there exists a constant $\varepsilon > 0$ such that for every $-\varepsilon < s < \varepsilon$,

$$\alpha \int_{\{\phi \neq 0\}} u(x, t-s) \frac{\partial \phi}{\partial t} d\nu + \int_{\{\phi \neq 0\}} g_{u(x,t-s)}^2 d\nu \leq K \int_{\{\phi \neq 0\}} g_{u(x,t-s)+\phi}^2 d\nu.$$

Let now $\zeta_\varepsilon(s)$ be a standard mollifier whose support is contained in $(-\varepsilon, \varepsilon)$. We multiply the above inequality with $\zeta_\varepsilon(s)$ and integrate on both sides with respect to s use Fubini's theorem to change the order of integration, and lastly use partial integration for the first term on the left hand side, to obtain

$$(2.6) \quad -\alpha \int_{\{\phi \neq 0\}} \frac{\partial u_\varepsilon}{\partial t} \phi d\nu + \int_{\{\phi \neq 0\}} (g_u^2)_\varepsilon d\nu \leq K \int_{\{\phi \neq 0\}} \left(g_{u(x,t-s)+\phi}^2 \right)_\varepsilon d\nu,$$

for every compactly supported $\phi \in \text{Lip}(\Omega_T)$. By Lemma 2.3 in [MMPP] we know the following density result: for every $\phi \in L^2(0, T; N^{1,2}(\Omega))$ and $\varepsilon > 0$ there exists a function $\varphi \in \text{Lip}(\Omega_T)$ such that $\{\varphi \neq 0\} \subset\subset \Omega_T$ and

$$\|\phi - \varphi\|_{L^2(0, T; N^{1,2}(\Omega))} < \varepsilon \quad \text{and} \quad \nu(\{\varphi \neq 0\} \setminus \{\phi \neq 0\}) < \varepsilon.$$

From this it follows, see the proof of Lemma 2.7 in [MMPP], that if $u \in L^2_{\text{loc}}(0, T; N^{1,2}(\Omega))$ is a K -quasiminimizer, then (2.6) holds for every $\phi \in L^2_c(0, T; N^{1,2}_0(\Omega))$.

2.8. Standing assumptions. We assume that the domain Ω is regular in the sense that $X \setminus \Omega$ is uniformly 2-thick. For the definition of thickness see below. Let $\eta : [0, T) \times \Omega \mapsto \mathbb{R}$ be such that $\eta \in W^{1,2}(0, T; N^{1,2}(\Omega))$, $\eta(x, 0) \in N^{1,2}(\Omega)$ and

$$\frac{1}{h} \int_0^h \int_{\Omega} |\eta(x, t) - \eta(x, 0)|^2 d\mu dt \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

From now on in this paper, we assume that $u \in L^2_{\text{loc}}(0, T; N^{1,2}(\Omega))$ is a parabolic quasiminimizer in Ω_T , and satisfies a parabolic boundary condition with η , in the sense that

$$(2.7) \quad u(\cdot, t) - \eta(\cdot, t) \in N^{1,2}_0(\Omega), \quad \text{for a.e. } t \in (0, T),$$

$$(2.8) \quad \frac{1}{h} \int_0^h \int_{\Omega} |u(x, t) - \eta(x, t)|^2 d\mu dt \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

3. ESTIMATES AWAY FROM THE LATERAL BOUNDARY

Establishing higher integrability for a function is based on obtaining a reverse Hölder inequality for the function, and then using it together with a Caldéron–Zygmund type decomposition and a Vitali covering to obtain integrability at some slightly higher exponent. The starting point for showing the reverse Hölder inequality for a parabolic quasiminimizer is an energy estimate over two concentric parabolic cylinders with different radii, $Q_{\rho}(z_0)$ and $Q_{\sigma}(z_0)$, where $\rho < \sigma$. This energy estimate is extracted from the definition of parabolic quasiminimizers by choosing a suitable test function.

When choosing the test function, we are faced with two qualitatively different situations. Depending on the center point and radii of the concentric cylinders, the larger cylinder $Q_{\sigma}(z_0) = B_{\sigma}(x_0) \times \Lambda_{\sigma}(t_0)$, may or may not overlap the lateral boundary of Ω_T . These two alternatives cause a difference in how we build the test function, and consequently lead to different energy estimates.

In case we assume $B_{\sigma}(x_0)$ is a subset of Ω , and so $Q_{\sigma}(z_0)$ does not overlap the lateral boundary of Ω_T , we can construct the test function by using only the geometry of the cylinders $Q_{\rho}(x_0)$ and $Q_{\sigma}(x_0)$, the quasiminimizer u itself, and the given initial condition, without having to take into consideration the lateral boundary of Ω_T .

We begin by treating this case. In order to establish the reverse Hölder inequality, it turns out that we only need the energy estimate for $\rho < \sigma \leq 2\rho$. Therefore the discussion in this section covers those cylinders $Q_\rho(x_0)$ for which we have $B_{2\rho}(x_0) \subset \Omega$. The complementary case to this covers the cylinders $Q_\rho(x_0)$ such that $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, and is the topic of Section 4.

Lemma 3.1 (Energy estimate). *There exists a positive constant $c = c(K)$, such that for every $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, $\rho < \sigma$ where $B_\sigma \subset \Omega$, we have*

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_\sigma(t)|^2 d\mu + \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ & \leq c \int_{(Q_\sigma \setminus Q_\rho) \cap \Omega_T} g_u^2 d\nu + \frac{c}{(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma \cap \Omega_T} |u - u_\sigma(t)|^2 d\nu \\ & \quad + c \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{B_\sigma} |\eta(x, 0) - \eta_\sigma(0)|^2 d\mu \end{aligned}$$

Proof. Assume $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, and $\rho < \sigma$ are such that $B_\sigma(x_0) \subset \Omega$ $\Lambda_\rho \cap (0, T) \neq \emptyset$. Assume $t' \in \Lambda_\rho \cap (0, T)$. Define

$$\chi_h(t) = \begin{cases} \frac{t-h}{h}, & h \leq t \leq 2h, \\ 1, & 2h \leq t \leq t' - 2h, \\ \frac{t'-h-t}{h}, & t' - 2h \leq t \leq t' - h, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi \in N^{1,2}(B_\sigma)$, $0 \leq \varphi \leq 1$, be such that $\varphi = 1$ in B_ρ , the support of φ is a compact subset of B_σ , and

$$g_\varphi^2 \leq \frac{c}{(\sigma - \rho)^2}.$$

For a function $f(x, t)$, denote

$$(3.1) \quad f_\sigma^\varphi(t) = \frac{\int_{B_\sigma} f(x, t) \varphi(x) d\mu}{\int_{B_\sigma} \varphi(x) d\mu}.$$

Set now $\phi = -\varphi(u_\varepsilon - (u_\varepsilon)_\sigma^\varphi)\chi_h$. Since $u \in L_{\text{loc}}^2(0, T; N^{1,2}(\Omega))$ is a parabolic quasiminimizer in Ω_T and $\phi \in \text{Lip}_c(0, T; N^{1,2}(\Omega))$, by the discussion in Section 2.7 we can insert ϕ into inequality (2.6) and examine the resulting terms. In the first term on the left hand side, we add and subtract $(u_\varepsilon)_\sigma^\varphi$ to obtain after integrating by parts

$$(3.2) \quad - \int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} \phi d\nu = \int_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t)) \frac{\partial \phi}{\partial t} d\nu - \int_{\Omega_T} \frac{\partial (u_\varepsilon)_\sigma^\varphi(t)}{\partial t} \phi d\nu.$$

Using the definition of $(u_\varepsilon)_\sigma^\varphi(t)$, we see that the last term on the right hand side vanishes

$$\begin{aligned} & \int_{\Omega_T} \frac{\partial(u_\varepsilon)_\sigma^\varphi(t)}{\partial t} \phi \, d\nu \\ &= - \int_0^{t'} \frac{\partial(u_{\sigma,\varepsilon}^\varphi(t))}{\partial t} \left(\int_{B_\sigma} u_\varepsilon \varphi \, d\mu - \frac{\int_{B_\sigma} \varphi \, d\mu \int_{B_\sigma} u_\varepsilon \varphi \, d\mu}{\int_{B_\sigma} \varphi \, d\mu} \right) \chi_h(t) \, dt = 0. \end{aligned}$$

Obtaining this vanishing property is one of the two reasons for defining the weighted average (3.1). The other reason is that the integral average of the function $|u - u_\sigma^\varphi|^2$ over B_σ is comparable to the integral average of $|u - u_\sigma|^2$ over B_σ , as will be seen at the end of the proof. We write out the first term on the right hand side of (3.2), and have

$$\begin{aligned} - \int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} \phi \, d\nu &= - \int_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))^2 \frac{\partial}{\partial t} (\varphi \chi_h) \, d\nu \\ &\quad - \frac{1}{2} \int_{\Omega_T} \frac{\partial}{\partial t} ((u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))^2) (\varphi \chi_h) \, d\nu \\ &= - \frac{1}{2} \int_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))^2 \frac{\partial}{\partial t} (\varphi \chi_h) \, d\nu, \end{aligned}$$

and so, taking into account the definition of χ_h , we arrive at

$$\begin{aligned} (3.3) \quad - \int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} \phi \, d\nu &= - \frac{1}{2h} \int_h^{2h} \int_{B_\sigma} |u_\varepsilon(x, t) - (u_\varepsilon)_\sigma^\varphi(t)|^2 \varphi(x) \, d\mu \\ &\quad + \frac{1}{2h} \int_{t'-2h}^{t'-h} \int_{B_\sigma} |u_\varepsilon(x, t) - (u_\varepsilon)_\sigma^\varphi(t)|^2 \varphi(x) \, d\mu. \end{aligned}$$

By the definition (3.1), we have for every $\varepsilon < h$

$$\begin{aligned} \int_{\{\phi \neq 0\}} (u_\sigma^\varphi - (u_\varepsilon)_\sigma^\varphi)^2 \, d\nu &\leq \mu(\Omega) \int_{h-\delta}^{T-h+\delta} (u_\delta^\varphi(t) - (u_\sigma^\varphi)_\varepsilon(t))^2 \, dt \\ &\leq \mu(\Omega) \int_{-\varepsilon}^\varepsilon \int_{h-\delta}^{T-h+\delta} |u_\sigma^\varphi(t) - u_\sigma^\varphi(t-s)|^2 \, dt \, \zeta_\varepsilon(s) \, ds, \end{aligned}$$

and therefore the fact that $u_\sigma^\varphi \in L_{\text{loc}}^2(0, T)$ implies that the above expression tends to zero as $\varepsilon \rightarrow 0$. Hence, using the triangle inequality and the initial condition (2.8) as first $\varepsilon \rightarrow 0$ and then $h \rightarrow 0$ leads us to

$$\begin{aligned} \lim_{\varepsilon, h \rightarrow 0} - \int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} \phi \, d\nu &= - \frac{1}{2} \int_{B_\sigma} |\eta(x, 0) - \eta_\sigma^\varphi(0)|^2 \varphi(x) \, d\mu \\ &\quad + \frac{1}{2} \int_{B_\sigma} |u(x, t') - u_\sigma^\varphi(t')|^2 \varphi(x) \, d\mu. \end{aligned}$$

On the right hand side of inequality (2.6), we note that for every h, ε , in the set $\{\phi \neq 0\}$ we have

$$\begin{aligned} (g_{u(\cdot, -s) - \varphi(u_\varepsilon - (u_\varepsilon)_\sigma^\varphi)\chi_h}^2)_\varepsilon &\leq c(g_{u(\cdot, -s) - u}^2)_\varepsilon + cg_{u - \varphi(u - u_\sigma^\varphi)}^2 \\ &\quad + cg_{\varphi(u - u_\sigma^\varphi)}^2(1 - \chi_h)^2 + cg_\varphi^2((u_\varepsilon)_\sigma^\varphi - u_\sigma^\varphi)^2\chi_h^2 \\ &\quad + c(u - u_\varepsilon)^2g_\varphi^2\chi_h^2 + c\varphi^2g_{u - u_\varepsilon}^2\chi_h^2. \end{aligned}$$

By Lemma 2.2 we know that $g_{u - u_\varepsilon}^2 \rightarrow 0$ and $(g_{u(\cdot, -s) - u}^2)_\varepsilon \rightarrow 0$ in $L_{\text{loc}}^1(\Omega_T)$ as $\varepsilon \rightarrow 0$. Hence, we obtain

$$\limsup_{\varepsilon, h \rightarrow 0} \int_{\{\phi \neq 0\}} \left(g_{u(\cdot, -s) + \phi}^2 \right)_\varepsilon d\nu \leq c \int_{Q_\sigma \cap \Omega_T} g_{u - \varphi(u - u_\sigma^\varphi)}^2 d\mu dt.$$

Now we note that since u_σ^φ does not depend on x and hence its upper gradient vanishes, we have

$$\int_{Q_\sigma \cap \Omega_T} g_{u - \varphi(u - u_\sigma^\varphi)}^2 d\nu \leq \int_{Q_\sigma \cap \Omega_T} |1 - \varphi|^2 g_u^2 d\nu + \int_{Q_\sigma \cap \Omega_T} |u - u_\sigma^\varphi(t)|^2 g_\varphi^2 d\nu.$$

Combining the obtained expressions leads us to the estimate

$$\begin{aligned} &\text{ess sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_\sigma^\varphi(t)|^2 d\mu + \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ &\leq c \int_{(Q_\sigma \setminus Q_\rho) \cap \Omega_T} g_u^2 d\nu + \frac{c}{(\sigma - \rho)^2} \int_{Q_\sigma \cap \Omega_T} |u - u_\sigma^\varphi(t)|^2 d\nu \\ &\quad + c \int_{B_\sigma} |\eta(x, 0) - \eta_\sigma^\varphi(0)|^2 d\mu, \end{aligned}$$

where $c = c(K)$. We complete the proof by noting that for any $t \in (0, T)$, we have

$$\begin{aligned} &\int_{B_\rho} |u - u_\sigma(t)|^2 d\mu \\ &\leq 2 \int_{B_\rho} |u - u_\sigma^\varphi(t)|^2 d\mu + 2 \int_{B_\rho} \left(\int_{B_\sigma} |u_\sigma^\varphi(t) - u|^2 d\mu \right) d\mu \\ &\leq 4 \int_{B_\rho} |u - u_\sigma^\varphi(t)|^2 d\mu. \end{aligned}$$

On the other hand, by the triangle inequality and by Jensen's inequality, and since $\varphi = 1$ in B_ρ ,

$$\begin{aligned} &\int_{B_\sigma} |u - u_\sigma^\varphi(t)|^2 d\mu \leq 2 \int_{B_\sigma} |u - u_\sigma(t)|^2 d\mu \\ &\quad + 2 \int_{B_\sigma} \left(\left(\int_{B_\sigma} \varphi d\mu \right)^{-1} \int_{B_\sigma} |u_\sigma(t) - u|^2 \varphi d\mu \right) d\mu \\ &\leq 4 \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{B_\sigma} |u - u_\sigma(t)|^2 d\mu. \end{aligned}$$

The analogous applies for the functions $\eta(x, 0)$, $\eta_\sigma(0)$ and $\eta_\sigma^\varphi(0)$. \square

Having established the fundamental energy estimate, we derive a Caccioppoli inequality by using the hole filling iteration. For the iteration, it is essential that we can write the above energy estimate for every $\sigma \in (\rho, 2\rho)$.

Lemma 3.2 (Caccioppoli). *There exists a positive constant $c = c(c_\mu, K)$ so that for any $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, such that $B_{2\rho}(x_0) \subset \Omega$, we have*

$$\int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \leq \frac{c}{\rho^2} \int_{Q_{2\rho} \cap \Omega_T} |u - u_{2\rho}(t)|^2 d\nu + c \int_{B_{2\rho}} |\eta(x, 0) - \eta_{2\rho}(0)|^2 d\mu.$$

Proof. By Lemma 3.1, for any cylinder $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $B_{2\rho}(x_0) \subset \Omega$, we have for any $\rho < \sigma \leq 2\rho$,

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in \Lambda_\rho(0, T)} \int_{B_\rho} |u - u_\sigma(t)|^2 d\mu + \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ & \leq c \int_{(Q_\sigma \setminus Q_\rho) \cap \Omega_T} g_u^2 d\nu + \frac{c}{(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma \cap \Omega_T} |u - u_\sigma(t)|^2 d\nu \\ & \quad + c \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{B_\sigma} |\eta(x, 0) - \eta_\sigma(x, 0)|^2 d\mu, \end{aligned}$$

where $c = c(K)$. We add $c \int_{Q_\rho} g_u^2 d\nu$ to both sides of the expression, and divide by $1 + c$, to obtain

$$\begin{aligned} \int_{Q_\rho} g_u^2 d\nu & \leq \frac{c}{1 + c} \int_{Q_\sigma} g_u^2 d\nu + \frac{c}{(1 + c)(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma} |u - u_\sigma(t)|^2 d\nu \\ & \quad + \frac{c}{(1 + c)} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{B_\sigma} |\eta(x, 0) - \eta_\sigma(x, 0)|^2 d\mu. \end{aligned}$$

Then we choose

$$\rho_0 = \rho, \quad \rho_i - \rho_{i-1} = \frac{1 - \beta}{\beta} \beta^i \rho, \quad i = 1, 2, \dots, k, \quad \beta^2 = \frac{1}{2} \left(\frac{c}{1 + c} + 1 \right),$$

replace ρ by ρ_{i-1} and σ by ρ_i , and iterate, to have

$$\begin{aligned} \int_{Q_\rho} g_u^2 d\nu & \leq \left(\frac{c}{1 + c} \right)^k \int_{Q_{\rho_k}} g_u^2 d\nu \\ & \quad + \sum_{i=1}^k \left(\frac{c}{1 + c} \right)^i \frac{\mu(B_{\rho_i})}{\mu(B_{\rho_{i-1}})} \left(\frac{1}{(\rho_i - \rho_{i-1})^2} \int_{Q_{\rho_i}} |u - u_{\rho_i}(t)|^2 d\nu \right. \\ & \quad \left. + \int_{B_{\rho_i}} |\eta(x, 0) - \eta_{\rho_i}(0)|^2 d\mu \right). \end{aligned}$$

Here among other things $\rho_i \leq 2\rho_{i-1}$ for every i , and so by the doubling property of μ , the ratio $\mu(B_{\rho_i})/\mu(B_{\rho_{i-1}})$ is uniformly bounded. Also, for

each i we can estimate after using Fubini's theorem,

$$\begin{aligned} \int_{Q_{\rho_i}} |u - u_{\rho_i}(t)|^2 d\nu &\leq 2 \int_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 d\nu \\ &+ 2 \int_{Q_{2\rho}} \int_{B_{\rho_i}} |u_{2\rho}(t) - u|^2 d\mu d\nu \leq 2c \int_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 d\nu, \end{aligned}$$

where $c = c(c_\mu)$, and analogously for η, η_{ρ_i} . Hence, taking the limit $k \rightarrow \infty$ yields the estimate,

$$\int_{Q_\rho} g_u^2 d\nu \leq \frac{c}{\rho^2} \int_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 d\nu + c \int_{B_{2\rho}} |\eta(x, 0) - \eta_{2\rho}(0)|^2 d\mu,$$

where $c = c(c_\mu, K)$. \square

Next we prove a parabolic version of the Poincaré inequality. We use the fundamental energy estimate with $\sigma = 2\rho$.

Lemma 3.3 (Parabolic Poincaré). *There exists positive constants $c = c(c_\mu, c_P, \lambda, K)$ and $1 < q_0 < 2$ so that for any $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, such that $B_{2\rho}(x_0) \subset \Omega$, we have*

$$\begin{aligned} \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_\rho(t)|^2 d\mu &\leq c \frac{\rho^2}{\nu(Q_{2\lambda\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ c\rho^2 \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}}, \end{aligned}$$

for any $q_0 \leq q$.

Proof. By Lemma 3.1

$$\begin{aligned} \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_{2\rho}|^2 d\mu &\leq c \int_{Q_{2\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ \frac{c}{\rho^2} \int_{Q_{2\rho} \cap \Omega_T} |u - u_{2\rho}|^2 d\nu + c \int_{B_{2\rho}} |\eta(x, 0) - \eta_{2\rho}(0)|^2 d\mu. \end{aligned}$$

Since by assumption $B_{2\rho} \subset \Omega$, we can use the $(2, 2)$ -Poincaré inequality for the second term on right hand side, and the $(2, q)$ -Poincaré inequality, where $1 < q < 2$ is as in Remark 2.1, for the third term on the right hand side. We obtain

$$\begin{aligned} \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_{2\rho}|^2 d\mu &\leq c \int_{\Lambda_\rho \cap (0, T)} \int_{B_{2\lambda\rho}} g_u^2 d\mu dt + c\rho^2 \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}}, \end{aligned}$$

where $c = c(c_\mu, c_P, K)$. The proof is completed by observing that $Q_{2\rho} = 4\rho^2\nu(B_{2\rho})$. \square

Caccioppoli's inequality together with the parabolic- and $(2, q)$ -Poincaré inequality now provide us the required tools to establish a reverse Hölder's inequality.

Lemma 3.4 (Reverse Hölder inequality). *There exists a positive constant $c = c(c_\mu, c_P, \lambda, K)$, and a $1 < q < 2$, so that for any $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, such that $B_{2\rho}(x_0) \subset \Omega$, we have*

$$\begin{aligned} \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu &\leq \varepsilon c \frac{1}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ \varepsilon^{-1} c \left(\frac{1}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^q d\nu \right)^{\frac{2}{q}} + \varepsilon c \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}}. \end{aligned}$$

Proof. By the Caccioppoli Lemma 3.2, by the doubling property of μ and since $\nu(Q_\rho) = \rho^2 \mu(B_\rho)$, and then the $(2, q)$ -Poincaré inequality for the second term on the right hand side, we obtain

$$\begin{aligned} \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu &\leq \frac{c}{\rho^4} \int_{\Lambda_\rho \cap \Omega_T} \int_{B_{2\rho}} |u - u_{2\rho}|^2 d\mu dt + c \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}}, \end{aligned}$$

where $c = c(c_\mu, c_P, K)$. On the other hand we can write

$$\begin{aligned} \frac{c}{\rho^4} \int_{\Lambda_\rho \cap \Omega_T} \int_{B_{2\rho}} |u - u_{2\rho}|^2 d\mu dt &\leq \left(\frac{c}{\rho^2} \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho} |u - u_{2\rho}|^2 d\mu \right)^{1 - \frac{q}{2}} \\ &\cdot \frac{c}{\rho^2} \int_{\Lambda_\rho \cap \Omega_T} \left(\frac{c}{\rho^2} \int_{B_{2\rho}} |u - u_{2\rho}|^2 d\mu \right)^{\frac{q}{2}} dt \\ &\leq \left\{ \frac{c}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^2 d\nu + c \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}} \right\}^{1 - \frac{q}{2}} \\ &\cdot \frac{c}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^q d\nu, \end{aligned}$$

where we used Lemma 3.3 and the $(2, q)$ -Poincaré inequality. By the ε -Young inequality we now obtain for every positive ε

$$\begin{aligned} \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu &\leq \varepsilon c \frac{1}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ \varepsilon^{-1} c \left(\frac{1}{\nu(Q_{2\rho})} \int_{Q_{2\lambda\rho} \cap \Omega_T} g_u^q d\nu \right)^{\frac{2}{q}} + \varepsilon c \left(\int_{B_{2\lambda\rho}} g_\eta^q(x, 0) d\mu \right)^{\frac{2}{q}}, \end{aligned}$$

where $c = c(c_\mu, c_P, \lambda, K)$. □

4. ESTIMATES NEAR THE LATERAL BOUNDARY

In this section we treat the almost complementary case to the one covered in section 3. This means that we establish a reverse Hölder estimate for parabolic quasiminimizers in the cylinders $Q_\rho(z_0) = B_\rho(x_0) \times \Lambda_\rho(t_0)$ which are such that $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$. However, in addition to this we will have to assume that ρ is small enough, and so we cover the situation where the cylinders are contained in the vicinity of the lateral boundary.

Continuing the discussion from the beginning of Section 3, here in the case where $Q_\sigma(z_0)$ may overlap the lateral boundary of Ω_T , we have to take the lateral boundary of Ω_T into consideration when building the test function for obtaining the energy estimate. Indeed, instead of relying solely on the geometry of $Q_\rho(z_0)$ and $Q_\sigma(z_0)$, we also make use of the lateral boundary condition.

After obtaining the energy estimate, as a consequence of building the lateral boundary condition into the test function, we cannot use the usual Poincaré inequality to the same extent as was done in section 3. Instead we use a version of the Poincaré inequality which introduces the variational capacity of the zero set of the function $u - \eta$.

Before going on, we introduce some concepts.

Definition 4.1. Let $F \subset X$ be an open set, and let $E \subset F$. The variational capacity is defined

$$\text{cap}_p(E, F) = \inf_f \int_F g_f^p d\mu,$$

where the infimum is taken over all $f \in N_0^{1,p}(F)$ such that $f \geq 1$ on E .

As can be seen from the following, in our setting the variational capacity is closely related to the measure of the sets.

Lemma 4.2. *Let X be a measure space equipped with a doubling measure μ , and satisfies a weak p -Poincaré inequality. Let $E \subset B_\rho(x)$ with $0 < \rho < (1/8)\text{diam}(X)$. Then there exists a positive constant $c = c(c_P, c_\mu, \lambda, p)$ such that*

$$\frac{\mu(E)}{c\rho^p} \leq \text{cap}_p(E, B_{2\rho}(x)) \leq c \frac{\mu(B_\rho(x))}{\rho^p}.$$

Proof. For proof we refer the reader to [Bj]. □

We will need the following version of Poincaré's inequality, which gives an upper gradient estimate for the integral average of any Newtonian function. We use the self improving property of the usual Poincaré inequality to obtain $1 < q < 2$ on the right hand side.

Theorem 4.3 (Poincaré with capacity). *Suppose $f \in N^{1,2}(B_{2\rho})$. Denote $N_{B_\rho}(f) = \{x \in B_\rho : f(x) = 0\}$. Then there exists a $1 < q_0 < 2$ and a*

positive constant $c = c(c_\mu, c_P, \lambda)$ such that for any $q_0 \leq q \leq 2$, we have

$$\left(\int_{B_{2\rho}} |f|^2 d\mu \right)^{\frac{1}{2}} \leq c \left(\frac{1}{\text{cap}_q(N_{B_\rho}(f), B_{2\rho})} \int_{B_{2\lambda\rho}} g_f^q d\mu \right)^{\frac{1}{q}},$$

for every $0 < \rho < (1/8)\text{diam}(X)$.

Proof. First we assume that

$$f_{B_{2\rho}} = \int_{B_{2\rho}} f(x) d\mu \neq 0.$$

Take $\phi \in \text{Lip}_c(B_{2\rho})$ and $0 \leq \phi \leq 1$, such that $\phi = 1$ in B_ρ and $g_\phi \leq \frac{2}{\rho}$. Define $v : X \rightarrow \mathbb{R}$ by setting

$$v = \begin{cases} \phi(f_{B_{2\rho}} - f) & \text{in } B_{2\rho} \\ 0 & \text{in } x \in X \setminus B_{2\rho}. \end{cases}$$

Then $v \in N^{1,2}(X)$, the support of v is a compact subset of $B_{2\rho}$ and we have $v = f_{B_{2\rho}} - f$ in B_ρ . From Remark 2.1 we know there exists a $1 < q_0 < 2$ so that for any $q_0 \leq q \leq 2$ the weak (q, q) -Poincaré inequality holds. By the product rule for upper gradients and then the (q, q) -Poincaré inequality,

$$\begin{aligned} \int_{B_{2\rho}} g_v^q d\mu &\leq \int_{B_{2\rho}} (g_f \phi + |f_{B_{2\rho}} - f| g_\phi)^q d\mu \\ &\leq c \int_{B_{2\rho}} g_f^q d\mu + c \frac{2^q}{\rho^q} \int_{B_{2\rho}} |f_{B_{2\rho}} - f|^q d\mu \leq c \int_{B_{\lambda 2\rho}} g_f^q d\mu, \end{aligned}$$

where $c = c(c_P)$. On the other hand, since $N_{B_\rho}(f) \subset \{f_{B_{2\rho}}^{-1} v = 1\}$, we have by the definition of the q -capacity

$$\frac{1}{|f_{B_{2\rho}}|^q} \int_{B_{2\rho}} g_v^q d\mu = \int_{B_{2\rho}} g_{f_{B_{2\rho}}^{-1} v}^q d\mu \geq \text{cap}_q(N_{B_\rho}(f), B_{2\rho}).$$

This gives us

$$|f_{B_{2\rho}}| \leq \left(\frac{1}{\text{cap}_q(N_{B_\rho}(f), B_{2\rho})} \int_{B_{2\rho}} g_v^q d\mu \right)^{\frac{1}{q}}.$$

Now we can use the $(2, q)$ -Poincaré inequality together with the above inequality, and then Lemma 4.2 to write

$$\begin{aligned} \left(\int_{B_{2\rho}} |f|^2 d\mu \right)^{\frac{1}{2}} &\leq \left(\int_{B_{2\rho}} |f_{B_{2\rho}} - f|^2 d\mu \right)^{\frac{1}{2}} + |f_{B_{2\rho}}| \\ &\leq c_P \rho \left(\int_{B_{2\lambda\rho}} g_f^q d\mu \right)^{\frac{1}{q}} + \left(\frac{1}{\text{cap}_q(N_{B_\rho}(f), B_{2\rho})} \int_{B_{2\rho}} g_v^q d\mu \right)^{\frac{1}{q}} \\ &\leq c \left(\frac{1}{\text{cap}_q(N_{B_\rho}(f), B_{2\rho})} \int_{B_{2\lambda\rho}} g_f^q d\mu \right)^{\frac{1}{q}}, \end{aligned}$$

for any $0 < \rho < (1/8)\text{diameter}(X)$, where $c = c(c_\mu, c_P, \lambda)$. Assume then that $f_{B_{2\rho}} = 0$. Then we may directly use the $(2, q)$ -Poincaré inequality together with Lemma 4.2 to obtain the result. \square

When considering the variational capacity of the zero set of $u - \eta$ in a metric ball overlapping the lateral boundary, the regularity of the lateral boundary of Ω_T in the sense of variational capacity comes into play. In order to build this into the assumption of the set Ω , we introduce the following.

Definition 4.4. A set $E \subset X$ is said to be uniformly p -thick, if there exist positive constants δ and ρ_0 so that

$$\text{cap}_p(E \cap B_\rho(x), B_{2\rho}(x)) \geq \delta \text{cap}_p(B_\rho(x), B_{2\rho}(x)),$$

for every $x \in E$ and $0 < \rho < \rho_0$.

The uniform p -thickness satisfies the following deep self improving property, which will be needed when showing the reverse Hölder inequality.

Theorem 4.5. *Let X be a proper linearly locally convex metric space endowed with a doubling regular Borel measure, supporting a $(1, q_0)$ -Poincaré inequality for some $1 \leq q_0 < \infty$. Let $p > q_0$ and suppose $E \subset X$ is uniformly p -thick. Then there exists $q < p$ so that E is uniformly q -thick.*

Proof. See [BMS]. \square

It is known, see Lemma 4.4 in [ATG], that a complete metric measure space equipped with a doubling measure is proper, and our assumptions for X are sufficient for using Theorem 4.5.

Now we can begin to build the estimates needed for the reverse Hölder inequality. As before, we start by choosing a convenient test function in the definition of parabolic quasiminimizers and derive the fundamental energy estimate. Notice that the following is a quite general estimate, we do not yet need the condition $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$.

Lemma 4.6 (Energy estimate). *There exists a positive constant $c = c(K)$, such that*

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho \cap \Omega} |u(x, t) - \eta(x, t)|^2 d\mu + \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ & \leq c \int_{(Q_\sigma \setminus Q_\rho) \cap \Omega_T} g_u^2 d\nu + c \int_{Q_\sigma \cap \Omega_T} |u - \eta|^2 \left(1 + \frac{1}{(\sigma - \rho)^2}\right) d\nu \\ & + c \int_{Q_\sigma \cap \Omega_T} \left(g_\eta^2 + \left|\frac{\partial \eta}{\partial t}\right|^2\right) d\nu. \end{aligned}$$

Proof. Assume $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $\Lambda_\rho(t_0) \cap (0, T) \neq \emptyset$ and $\rho < \sigma$. Let $t' \in \Lambda_\rho \cap (0, T)$, and define

$$\chi_h(t) = \begin{cases} \frac{t-h}{h}, & h \leq t \leq 2h, \\ 1, & 2h \leq t \leq t' - 2h, \\ \frac{t'-h-t}{h}, & t' - 2h \leq t \leq t' - h, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi \in C^\infty(0, T; N_0^{1,2}(B_\sigma))$, $0 \leq \varphi \leq 1$, be such that $\varphi = 1$ in B_ρ , and

$$(4.1) \quad g_\varphi^2 + \left|\frac{\partial \varphi}{\partial t}\right| \leq \frac{2}{(\sigma - \rho)^2}.$$

Consider the function $\phi = -\varphi(u_\varepsilon - \eta_\varepsilon)\chi_{h,\delta}$. Again, since u is a parabolic quasiminimizer in Ω_T and ϕ has the required smoothness for a test function, we can insert ϕ in inequality (2.6) and examine the resulting terms. We begin by examining the first term on the left hand side. After adding and subtracting η_ε and then conducting partial integration with respect to time, we can write

$$\begin{aligned} & - \int_{\{\phi \neq 0\}} \frac{\partial u_\varepsilon}{\partial t} \phi d\nu \\ & = \int_{\{\phi \neq 0\}} \frac{1}{2} \frac{\partial}{\partial t} ((u_\varepsilon - \eta_\varepsilon)^2) \varphi \chi_h d\nu + \int_{\{\phi \neq 0\}} \frac{\partial \eta_\varepsilon}{\partial t} \varphi (u_\varepsilon - \eta_\varepsilon) \chi_h d\nu. \end{aligned}$$

Performing partial integration on the first term on the right hand side yields now

$$\begin{aligned} & - \int_{\{\phi \neq 0\}} \frac{\partial u_\varepsilon}{\partial t} \phi d\nu \\ & = -\frac{1}{2h} \int_h^{2h} \int_{B_\sigma \cap \Omega} (u_\varepsilon - \eta_\varepsilon)^2 \varphi d\mu dt + \frac{1}{2h} \int_{t'-2h}^{t'-h} \int_{B_\sigma \cap \Omega} (u_\varepsilon - \eta_\varepsilon)^2 \varphi d\mu dt \\ & - \int_{\{\phi \neq 0\}} \frac{1}{2} (u_\varepsilon - \eta_\varepsilon)^2 \frac{\partial \varphi}{\partial t} \chi_h d\nu - \int_{\{\phi \neq 0\}} \frac{\partial \eta_\varepsilon}{\partial t} \varphi (u_\varepsilon - \eta_\varepsilon) \chi_h d\nu. \end{aligned}$$

Hence, after taking the limit $\varepsilon \rightarrow 0$ and then the limit $h \rightarrow 0$ we have the following: For almost every $0 < t' < T$, using the initial condition (2.8)

yields

$$\begin{aligned} \lim_{\varepsilon, h \rightarrow 0} - \int_{\{\phi \neq 0\}} u_\varepsilon \frac{\partial \phi}{\partial t} d\nu &\geq \frac{1}{2} \int_{B_\rho \cap \Omega} (u(x, t') - \eta(x, t'))^2 \varphi(x, t') d\mu \\ &\quad - \int_{Q_\sigma \cap \Omega_T} \frac{1}{2} (u - \eta)^2 \left| \frac{\partial \varphi}{\partial t} \right| d\nu - \int_{Q_\sigma \cap \Omega_T} \left| \frac{\partial \eta}{\partial t} \right| \varphi |u - \eta| d\nu. \end{aligned}$$

Also,

$$\lim_{\varepsilon, h \rightarrow 0} \int_{\{\phi \neq 0\}} (g_u^2)_\varepsilon d\nu \geq \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu.$$

On the right hand side of (2.6), $g_{(u-\eta)-(u-\eta)_\varepsilon}^2 \rightarrow 0$ and $(g_{u(\cdot, \cdot-s)-u}^2)_\varepsilon \rightarrow 0$ in $L_{\text{loc}}^1(\Omega_T)$ as $\varepsilon \rightarrow 0$. Hence

$$\begin{aligned} \limsup_{\varepsilon, h \rightarrow 0} \int_{\{\phi \neq 0\}} \left(g_{u(\cdot, \cdot-s)-\phi}^2 \right)_\varepsilon d\nu &\leq c \int_{Q_\sigma \cap \Omega_T} g_{u-\varphi(u-\eta)}^2 d\nu \\ &\leq c \int_{Q_\sigma \cap \Omega_T} (g_{(1-\varphi)(u-\eta)}^2 + g_\eta^2) d\nu \leq c \int_{Q_\sigma \cap \Omega_T} (1-\varphi)^2 (g_u^2 + g_\eta^2) d\nu \\ &\quad + c \int_{Q_\sigma \cap \Omega_T} |u - \eta|^2 g_\varphi^2 d\nu + c \int_{Q_\sigma \cap \Omega_T} g_\eta^2 d\nu. \end{aligned}$$

Noting that $\varphi = 1$ in Q_ρ , combining all the obtained results together through (2.6) and then using Young's inequality and (4.1) yields us the desired expression. \square

Having established the fundamental energy estimate, we derive from it a Caccioppoli inequality by using the hole filling iteration. As in section 3, we use the fundamental energy estimate for $\rho < \sigma < 2\rho$.

Theorem 4.7 (Caccioppoli). *There exists a positive constant $c = c(K)$, such that*

$$\int_{Q_\rho \cap \Omega} g_u^2 d\nu \leq c \int_{Q_{2\rho} \cap \Omega} |u - \eta|^2 \left(1 + \frac{1}{\rho^2} \right) d\nu + c \int_{Q_{2\rho} \cap \Omega} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu.$$

Proof. After adding $c \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu$ to both sides of the expression in Lemma 4.6, and then dividing by $c + 1$, we can write

$$\begin{aligned} &\int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ &\leq \frac{c}{c+1} \int_{(Q_\sigma \setminus Q_\rho) \cap \Omega_T} g_u^2 d\nu + \frac{c}{c+1} \int_{Q_\sigma \cap \Omega_T} |u - \eta|^2 \left(1 + \frac{2}{(\sigma - \rho)^2} \right) d\nu \\ &\quad + \frac{c}{c+1} \int_{Q_\sigma \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu. \end{aligned}$$

Then we choose

$$\rho_0 = \rho, \quad \rho_i - \rho_{i-1} = \frac{1-s}{s} s^i \rho, \quad i = 1, 2, \dots, k \quad s^2 = \frac{1}{2} \left(\frac{c}{c+1} + 1 \right),$$

replace ρ by ρ_{i-1} and σ by ρ_i , and iterate to obtain

$$\begin{aligned} \int_{Q_\rho \cap \Omega_T} g_u^2 d\mu dt &\leq \left(\frac{c}{c+1}\right)^k \int_{Q_{\rho_k} \cap \Omega_T} g_u^2 d\nu \\ &+ \sum_{i=1}^k \left(\frac{c}{c+1}\right)^i \left(\int_{Q_{\rho_i} \cap \Omega_T} |u - \eta|^2 \left(1 + \frac{2}{(\rho_i - \rho_{i-1})^2}\right) d\nu \right. \\ &\left. + \int_{Q_{\rho_i} \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu \right). \end{aligned}$$

Now, taking the limit $k \rightarrow \infty$ leads to the expression

$$\int_{Q_\rho \cap \Omega} g_u^2 d\nu \leq c \int_{Q_{2\rho} \cap \Omega} |u - \eta|^2 \left(1 + \frac{1}{\rho^2}\right) d\nu + c \int_{Q_{2\rho} \cap \Omega} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu,$$

where $c = c(K)$. \square

Then we prove a parabolic version of the Poincaré inequality for the function $u - \eta$, in the vicinity of the lateral boundary. We use the energy estimate with $\sigma = 2\rho$. This is the stage at which we need the assumption that x_0 and ρ are such that $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, and moreover we need to be close enough to the lateral boundary, i.e. that ρ is small enough. These assumptions enable us to exploit the uniform thickness of Ω to obtain an upper gradient estimate for the integral average of $u - \eta$.

Theorem 4.8 (Parabolic Poincaré). *Assume $X \setminus \Omega$ is uniformly 2-thick. Then there exist a positive constant $\rho_0 < (1/8)\text{diam}(X)$ and a positive constant $c = c(c_\mu, c_P, \lambda, K)$, such that for every $0 < \rho < \rho_0$ and parabolic cylinder $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, we have*

$$\text{ess sup}_{t \in \Lambda_\rho \cap (0, T)} \int_{B_\rho \cap \Omega} |u - \eta|^2 d\mu \leq c \int_{Q_{6\lambda\rho} \cap \Omega_T} \left(g_u^2 + g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu.$$

Proof. Assume a cylinder $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, where $x_0 \in X$ is such that $B_{2\lambda\rho}(x_0) \setminus \Omega \neq \emptyset$. From Lemma 4.6, we have

$$\begin{aligned} \text{ess sup}_{\Lambda_\rho \cap (0, T)} \int_{B_\rho \cap \Omega} |u - \eta|^2 d\mu &\leq c \int_{Q_{2\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ c \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 \left(1 + \frac{2}{\rho^2}\right) d\nu + c \int_{Q_{2\rho} \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu, \end{aligned}$$

where $c = c(K)$. For $0 < \rho < M$, we can estimate $1 \leq M^2/\rho^2$ on the right hand side to obtain

$$\begin{aligned} (4.2) \quad \text{ess sup}_{\Lambda_\rho \cap (0, T)} \int_{B_\rho \cap \Omega} |u - \eta|^2 d\mu &\leq c \int_{Q_{2\rho} \cap \Omega_T} g_u^2 d\nu \\ &+ \frac{c}{\rho^2} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu + c \int_{Q_{2\rho} \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu, \end{aligned}$$

where $c = c(K, M)$. Next we continue the mapping $u(\cdot, t) - \eta(\cdot, t)$ outside of Ω by setting $u(\cdot, t) - \eta(\cdot, t) = 0$ in $X \setminus \Omega$. By assumption $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, and so there exists a point $x' \in X \setminus \Omega$, such that $B_{2\rho}(x_0) \subset B_{4\rho}(x')$ and $B_{4\lambda\rho}(x') \subset B_{6\lambda\rho}(x_0)$. Since $X \setminus \Omega$ is uniformly 2-thick, and then by Lemma 4.2, there exist positive constants $\rho_0 < (1/8)\text{diameter}(X)$ and $c = c(c_P, c_\mu)$, such that

$$\begin{aligned} \text{cap}_2(N_{B_{2\rho}(x')}(u - \eta), B_{4\rho}(x')) &\geq \text{cap}_2((X \setminus \Omega) \cap B_{2\rho}(x'), B_{4\rho}(x')) \\ &\geq \delta \text{cap}_2(B_{2\rho}(x'), B_{4\rho}(x')) \geq \delta c \frac{\mu(B_{2\rho}(x'))}{\rho^2}, \end{aligned}$$

for every $0 < \rho < \rho_0$. Hence, after using Lemma 4.3, we can estimate that for any $0 < \rho < \rho_0$

$$\begin{aligned} \frac{1}{\rho^2} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu &\leq \frac{1}{\rho^2} \int_{\Lambda_{2\rho} \cap (0, T)} \mu(B_{4\rho}(x')) \int_{B_{4\rho}(x')} |u - \eta|^2 d\mu dt \\ &\leq \frac{c}{\rho^2} \int_{\Lambda_{2\rho} \cap (0, T)} \frac{\mu(B_{4\rho}(x'))}{\text{cap}_2(N_{B_{2\rho}(x')}(u - \eta), B_{4\rho}(x'))} \int_{B_{4\lambda\rho}(x')} g_{u-\eta}^2 d\mu dt \\ &\leq c \int_{Q_{6\lambda\rho} \cap \Omega_T} g_u^2 d\nu + c \int_{Q_{6\lambda\rho} \cap \Omega_T} g_\eta^2 d\nu, \end{aligned}$$

where $c = c(c_\mu, c_P, \delta)$. Here we also used the fact that $g_{u-\eta}(\cdot, t) = 0$ μ -almost everywhere outside of Ω . Plugging this into (4.2) completes the proof. \square

Now we start from the Caccioppoli inequality, and then combine the parabolic Poincaré inequality together with the (2,q)-Poincaré with capacity to obtain a reverse Hölder inequality. We use the self improving property of the uniform thickness to have control over the variational capacity of the zero set of $u - \eta$. Again, we need the assumption that we are close enough to the lateral boundary, in other words that ρ is small enough and $B_\rho(x_0) \setminus \Omega \neq \emptyset$.

Theorem 4.9 (Reverse Hölder inequality). *Suppose that $X \setminus \Omega$ is uniformly 2-thick. Then there exist positive constants $\rho_0 < (1/8)\text{diameter}(X)$, $c = c(c_\mu, c_P, \lambda, K)$ and $1 < q_0 < 2$ such that for every $0 < \rho < \rho_0$ and $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, we have*

$$\begin{aligned} \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu &\leq \varepsilon \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} g_u^2 d\nu + \varepsilon^{-1} c \left(\frac{1}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} g_u^q d\nu \right)^{\frac{2}{q}} \\ &\quad + (\varepsilon^{-1} + \varepsilon) \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu, \end{aligned}$$

for any positive ε and $q_0 \leq q$.

Proof. Let $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, such that $B_{2\lambda\rho}(x_0) \setminus \Omega \neq \emptyset$. From Theorem 4.7 we know that for every $0 < \rho < M$, we have

$$\begin{aligned} & \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ & \leq \frac{c}{\rho^2 \nu(Q_{2\rho})} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu + \frac{c}{\nu(Q_{2\rho})} \int_{Q_\rho \cap \Omega_T} \left(\left| \frac{\partial \eta}{\partial t} \right|^2 + g_\eta^2 \right) d\nu, \end{aligned}$$

where $c = c(K, M)$. Similarly to what was done earlier, for almost every $t \in (0, T)$, we continue the mapping $u(\cdot, t) - \eta(\cdot, t)$ to be zero outside Ω , and so from now on the mapping $u(\cdot, t) - \eta(\cdot, t)$ is to be thought of as defined in the whole space X . Also, since by assumption $B_{2\rho}(x_0) \setminus \Omega \neq \emptyset$, there exists a point $x' \in X \setminus \Omega$, such that $B_{2\rho}(x_0) \subset B_{4\rho}(x')$, and $B_{4\lambda\rho}(x') \subset B_{6\lambda\rho}(x_0)$. Let $1 < q < 2$ be as in Lemma 4.3. We have

$$\begin{aligned} & \frac{1}{\rho^2 \nu(Q_{2\rho})} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu \\ & \leq \frac{1}{\rho^4} \int_{\Lambda_{2\rho} \cap (0, T)} \left(\int_{B_{2\rho}} |u - \eta|^2 d\mu \right)^{1-\frac{q}{2}} \left(\int_{B_{2\rho}} |u - \eta|^2 d\mu \right)^{\frac{q}{2}} dt \\ & \leq \frac{c}{\rho^4} \left(\operatorname{ess\,sup}_{\Lambda_{2\rho} \cap (0, T)} \int_{B_{2\rho}} |u - \eta|^2 d\mu \right)^{1-\frac{q}{2}} \int_{\Lambda_{2\rho} \cap (0, T)} \left(\int_{B_{4\rho}(x')} |u - \eta|^2 d\mu \right)^{\frac{q}{2}} dt, \end{aligned}$$

where $c = c(c_\mu, \lambda)$. In the above expression the former factor on the right hand side can be estimated by Theorem 4.8, and the latter factor by Theorem 4.3. We obtain that for some $1 < q_0 < 2$, we have for every $q_0 \leq q \leq 2$ and $0 < 2\rho < (1/8)\operatorname{diameter}(X)$,

$$\begin{aligned} & \frac{1}{\rho^2 \nu(Q_{2\rho})} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu \\ & \leq \frac{c}{\rho^2} \left\{ \frac{\rho^2}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} \left(g_u^2 + g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu \right\}^{1-\frac{q}{2}} \\ & \quad \cdot \int_{\Lambda_{2\rho} \cap (0, T)} \frac{1}{\operatorname{cap}_q(N_{B_{2\rho}(x')}(u - \eta), B_{4\rho}(x'))} \int_{B_{4\lambda\rho}(x')} g_{u-\eta}^q d\mu dt, \end{aligned}$$

where $c = c(c_\mu, c_P, K)$. By assumption, $X \setminus \Omega$ is uniformly 2-thick. By Theorem 4.5 this implies that for some $1 < q < 2$ the set $X \setminus \Omega$ is also uniformly q -thick with some positive constant δ . We use this with Lemma 4.2, to conclude that

$$\begin{aligned} \operatorname{cap}_q(N_{B_{2\rho}(x')}(u - \eta), B_{4\rho}(x')) & \geq \operatorname{cap}_q(X \setminus \Omega \cap B_{2\rho}(x'), B_{4\rho}(x')) \\ & \geq \delta \operatorname{cap}_q(B_{2\rho}(x'), B_{4\rho}(x')) \\ & \geq \delta c \frac{\mu(B_{2\rho}(x'))}{\rho^q}, \end{aligned}$$

for every $0 < \rho < \rho_0 < (1/8)\text{diameter}(X)$, where $c = c(c_\mu, c_P)$. Hence for every $\rho < \rho_0$, we obtain

$$\begin{aligned} & \frac{c}{\rho^2 \nu(Q_{2\rho})} \int_{Q_{2\rho} \cap \Omega_T} |u - \eta|^2 d\nu \\ & \leq \left\{ \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} \left(g_u^2 + g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu \right\}^{1-\frac{q}{2}} \\ & \quad \cdot \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} g_{u-\eta}^q d\nu. \end{aligned}$$

Now we can use the ε -Young inequality and then Hölder's inequality to conclude that for every positive ε we have the estimate

$$\begin{aligned} & \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g_u^2 d\nu \\ & \leq \varepsilon \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} g_u^2 d\nu + \varepsilon^{-1} c \left(\frac{1}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} g_u^q d\nu \right)^{\frac{2}{q}} \\ & \quad + (\varepsilon^{-1} + \varepsilon) \frac{c}{\nu(Q_{6\lambda\rho})} \int_{Q_{6\lambda\rho} \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu. \end{aligned}$$

where the positive constant $c = c(c_\mu, c_P, \lambda, K)$. \square

5. GLOBAL HIGHER INTEGRABILITY

In this section we cover the steps from the reverse Hölder inequality both near and away from the lateral boundary of Ω_T , to the global higher integrability of a parabolic quasiminimizer's upper gradient.

We begin by proving a modification of Gehring's Lemma in metric spaces, to take into account the terms in the reverse Hölder inequalities that result from the initial and lateral boundary conditions.

In the proof of this theorem the initial cylinder is divided into a good set where g is bounded and into a bad set where g is unbounded. At each point of the bad set, in some small enough cylinder centered at this point, we have by our previous results a reverse Hölder inequality. These cylinders are then used to form a Vitali covering of the bad set, so that we obtain the reverse Hölder inequality over to whole bad set. Finally, by an argument involving Fubini's theorem, the reverse Hölder inequality is used to establish higher integrability over the bad set.

Theorem 5.1. *Let $g \in L_{loc}^2(0, T; N_{loc}^{1,2}(\Omega))$ and let f_1, f_2 be non negative measurable functions defined in Ω_T . Consider a parabolic cylinder $Q_{2R}(z_0) = B_{2R}(x_0) \times \Lambda_{2R}(t_0) \subset X \times \mathbb{R}$. Let s be the constant from (2.1) and let q be such that $2s/(2+s) < q < 2$. Suppose that there exists a positive constant $A > 1$, for which with any $z' = (x', t')$ and ρ such that*

$Q_{A\rho}(z') \subset Q_{2R}(z_0)$, we have, after abbreviating $Q_\rho = Q_\rho(z')$, $Q_{A\rho} = Q_{A\rho}(z')$ and $B_{A\rho} = B_{A\rho}(x')$,

$$(5.1) \quad \begin{aligned} & \frac{1}{\nu(Q_\rho)} \int_{Q_\rho \cap \Omega_T} g^2 d\nu \leq \varepsilon \frac{1}{\nu(Q_{A\rho})} \int_{Q_{A\rho} \cap \Omega_T} g^2 d\nu \\ & + \gamma \left(\frac{1}{\nu(Q_{A\rho})} \int_{Q_{A\rho} \cap \Omega_T} g^q d\nu \right)^{2/q} + \gamma \frac{1}{\nu(Q_{A\rho})} \int_{Q_{A\rho} \cap \Omega_T} f_1^2 d\nu \\ & + \gamma \left(\frac{1}{\mu(B_{A\rho})} \int_{B_{A\rho} \cap \Omega} f_2^q d\mu \right)^{2/q}, \end{aligned}$$

for any $\varepsilon > 0$, where γ may depend on ε . Then there exists positive constants $\varepsilon_0 = \varepsilon_0(c_\mu, A, \gamma, q)$ and $c = c(c_\mu, A, \gamma)$, such that

$$\begin{aligned} & \left(\frac{1}{\nu(Q_R)} \int_{Q_R \cap \Omega_T} g^{2+\varepsilon} d\nu \right)^{\frac{1}{2+\varepsilon}} \leq \left(\frac{c}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} g^2 d\nu \right)^{\frac{1}{2}} \\ & + \left(\frac{c}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} f_1^{2+\varepsilon} d\nu \right)^{\frac{1}{2+\varepsilon}} + \left(\frac{c}{\mu(B_{2R})} \int_{B_{2R} \cap \Omega} f_2^{q+\varepsilon} d\mu \right)^{\frac{1}{q+\varepsilon}}, \end{aligned}$$

for every $0 < \varepsilon \leq \varepsilon_0$, where we have abbreviated $Q_R = Q_R(z_0)$, $Q_{2R} = Q_{2R}(z_0)$ and $B_{2R} = B_{2R}(x_0)$.

Proof. Assume a parabolic cylinder Q_{2R} with center point $z_0 = (x_0, t_0)$. Define for every $z_1 = (x_1, x_2)$, $z_2 = (x_2, t_2) \in X \times \mathbb{R}$ the parabolic distance

$$\text{dist}_p(z_1, z_2) = d(x_1, x_2) + |t_1 - t_2|^{1/2}.$$

Using this, set for every $z \in Q_{2R}$ the functions

$$\begin{aligned} r(z) &= \text{dist}_p(z, (X \times \mathbb{R}) \setminus Q_{2R}), \\ \alpha(z) &= \frac{\nu(Q_{2R})}{\nu(Q_{\frac{r(z)}{5A}}(z))}. \end{aligned}$$

From the definition of $r(z)$ it can readily be checked that $Q_{r(z)}(z) \subset Q_{2R}$ for every $z \in Q_{2R}$. For $z \in Q_{2R}$, define

$$h(z) = \alpha^{-1/2}(z)g(z),$$

and for every $\lambda > 0$, set

$$G(\lambda) = \{ z \in Q_{2R} \cap \Omega_T : h(z) > \lambda \}.$$

Denote

$$\lambda_0 = \left(\frac{1}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} g^2 d\nu \right)^{1/2}.$$

Assume $\lambda > \lambda_0$. For ν -almost every $z' \in G(\lambda)$, we have for every $r \in [r(z')/(5A), r(z')]$,

$$(5.2) \quad \frac{1}{\nu(Q_r(z'))} \int_{Q_r(z') \cap \Omega_T} g^2 d\nu \leq \frac{\alpha(z')}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} g^2 d\nu \leq \alpha(z') \lambda^2,$$

and by the definition of $G(\lambda)$, since μ is a positive Borel measure,

$$(5.3) \quad \lim_{r \rightarrow 0} \frac{1}{\nu(Q_r(z'))} \int_{Q_r(z') \cap \Omega_T} g^2 d\nu = g^2(z') > \alpha(z') \lambda^2.$$

Now (5.2) and (5.3) imply that for ν -almost every $z' \in G(\lambda)$, there exists a corresponding radius $\rho(z') \in (0, r(z')/(5A))$, for which it holds

$$(5.4) \quad \begin{aligned} & \frac{1}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} g^2 d\nu \\ & \leq \alpha(z') \lambda^2 \leq \frac{1}{\nu(Q_{\rho(z')}(z'))} \int_{Q_{\rho(z')}(z') \cap \Omega_T} g^2 d\nu. \end{aligned}$$

Thus by choosing $\varepsilon = 1/2$ in (5.1), we can absorb the first term on the right hand side of (5.1) into the left hand side and obtain

$$\begin{aligned} & \frac{1}{\nu(Q_{\rho(z')}(z'))} \int_{Q_{\rho(z')}(z') \cap \Omega_T} g^2 d\nu \leq \left(\frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} g^q d\nu \right)^{2/q} \\ & \quad + \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} f_1^2 d\nu \\ & \quad + \left(\frac{c}{\mu(B_{A\rho(z')}(z'))} \int_{B_{A\rho(z')}(z') \cap \Omega} f_2^q d\mu \right)^{2/q}, \end{aligned}$$

for ν -almost every $z' \in G(\lambda)$, where $c = c(\gamma)$. This together with (5.4) yields

$$(5.5) \quad \begin{aligned} & \frac{1}{\nu(Q_{5A\rho(z')}(z'))} \int_{Q_{5A\rho(z')}(z') \cap \Omega_T} g^2 d\nu \\ & \leq \left(\frac{c}{\mu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} g^q d\nu \right)^{2/q} \\ & \quad + \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} f_1^2 d\nu \\ & \quad + \left(\frac{c}{\mu(B_{A\rho(z')}(z'))} \int_{B_{A\rho(z')}(z') \cap \Omega} f_2^q d\mu \right)^{2/q}, \end{aligned}$$

where $c = c(A, c_\mu, \gamma)$. From the definitions of a parabolic cylinder and the parabolic distance, it follows that

$$2^{-1/2}r(z') \leq r(z) \leq 2r(z') \quad \text{for every } z \in Q_{r(z')}(z'), \quad z' \in Q_{2R}.$$

From this it is straightforward to check that

$$\begin{aligned} Q_{r(z)}(z) &\subset Q_{3r(z')}(z'), & \text{for every } z \in Q_{r(z')}(z'), \ z' \in Q_{2R}, \\ Q_{r(z')}(z') &\subset Q_{4r(z)}(z) \end{aligned}$$

and so by the doubling property of the measure there exists positive constants $c = c(c_\mu)$, $c' = c'(c_\mu)$ such that

$$(5.6) \quad c\alpha(z') \leq \alpha(z) \leq c'\alpha(z) \quad \text{for every } z \in Q_{r(z')}(z'), \ z' \in Q_{2R}.$$

Because of this, we see from (5.5) that there exists a positive constant $c = c(A, c_\mu, \gamma)$, such that for ν -almost every $z' \in G(\lambda)$, after also using the fact that $\alpha(z) \geq 1$,

$$\begin{aligned} (5.7) \quad & \frac{1}{\nu(Q_{5A\rho(z')}(z'))} \int_{Q_{5A\rho(z')}(z') \cap \Omega_T} h^2 d\nu \\ & \leq \left(\frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} h^q d\nu \right)^{2/q} \\ & \quad + \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} f_1^2 d\nu \\ & \quad + \left(\frac{c}{\mu(B_{A\rho(z')}(z'))} \int_{B_{A\rho(z')}(z') \cap \Omega} f_2^q d\mu \right)^{2/q}. \end{aligned}$$

On the other hand, by Hölder's inequality since $1 < q < 2$, and then by (5.6), we obtain from (5.4),

$$\begin{aligned} (5.8) \quad & \left(\frac{1}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} h^q d\nu \right)^{(2-q)/q} \\ & \leq \left(\frac{1}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap \Omega_T} h^2 d\nu \right)^{(2-q)/2} \leq c\lambda^{2-q}, \end{aligned}$$

where $c = c(c_\mu)$. Define

$$\begin{aligned} G_{f_1}(\lambda) &= \{z \in Q_{2R} \cap \Omega_T : f_1 > \lambda\}, \\ G_{f_2}(\lambda) &= \{z \in B_{2R} \cap \Omega : f_2 > \lambda\}. \end{aligned}$$

Assume now any $\delta > 0$. By (5.7) and by the definitions of $G(\delta\lambda)$, $G_{f_1}(\delta\lambda)$ and $G_{f_2}(\delta\lambda)$, we have for ν -almost every $z' \in G(\lambda)$,

$$\begin{aligned} & \frac{1}{\nu(Q_{5A\rho(z')}(z'))} \int_{Q_{5A\rho(z')}(z') \cap \Omega_T} h^2 d\nu \\ & \leq c\delta^2\lambda^2 + \left(\frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap G(\delta\lambda)} h^q d\nu \right)^{2/q} \\ & \quad + \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap G_{f_1}(\delta\lambda)} f_1^2 d\nu \\ & \quad + \left(\frac{c}{\mu(B_{A\rho(z')}(z'))} \int_{B_{A\rho(z')}(z') \cap G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q}. \end{aligned}$$

By (5.6) and (5.4), we can now choose a small enough positive number $\delta(c_\mu, A, \gamma) < 1$ to absorb the first term on the right hand side into the left hand side. We obtain a positive $c = c(A, c_\mu, \gamma)$, such that for ν -almost every $z' \in G(\lambda)$ and any $\lambda > \lambda_0$, after using (5.8),

$$\begin{aligned} & \frac{1}{\nu(Q_{5A\rho(z')}(z'))} \int_{Q_{5A\rho(z')}(z')} h^2 d\nu \\ & \leq \lambda^{2-q} \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap G(\delta\lambda)} h^q d\nu \\ (5.9) \quad & \quad + \frac{c}{\nu(Q_{A\rho(z')}(z'))} \int_{Q_{A\rho(z')}(z') \cap G_{f_1}(\delta\lambda)} f_1^2 d\nu \\ & \quad + \left(\frac{c}{\mu(B_{A\rho(z')}(z'))} \int_{B_{A\rho(z')}(z') \cap G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q}. \end{aligned}$$

The collection $\{Q_{A\rho(z')}(z') : z' \in G(\lambda)\}$ is now an open cover of $G(\lambda)$. By the Vitali covering lemma, there exists a countable and pairwise disjoint subcollection $\{Q_{A\rho(z'_i)}(z'_i) : z'_i \in G(\lambda)\}_{i=1}^\infty$, such that

$$G(\lambda) \subset \bigcup_{i=1}^\infty Q_{5A\rho(z'_i)}(z'_i) \subset Q_{2R}.$$

The last inclusion follows from the fact that $5A\rho(z) \leq r(z)$. This property is the reason why we introduced the number 5 into the proof earlier. Now we can write for any $\lambda > \lambda_0$, after multiplying inequality (5.9) with

$\nu(Q_{A\rho(z')}(z'))$ and using the doubling property of μ ,

$$\begin{aligned} \int_{G(\lambda)} h^2 d\nu &\leq \sum_{i=1}^{\infty} \int_{Q_{5A\rho(z'_i)}(z'_i)} h^2 d\nu \\ &\leq \sum_{i=1}^{\infty} \left(c\lambda^{2-q} \int_{Q_{A\rho(z'_i)}(z'_i) \cap G(\delta\lambda)} h^q d\nu + c \int_{Q_{A\rho(z'_i)}(z'_i) \cap G_{f_1}(\delta\lambda)} f_1^2 d\nu \right. \\ &\quad \left. + c \frac{(\rho(z'_i))^2 \mu(B_{\rho(z'_i)}(z'_i))}{\mu(B_{\rho(z'_i)}(z'_i))^{2/q}} \left(\int_{B_{A\rho(z'_i)}(z'_i) \cap G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q} \right), \end{aligned}$$

where $c = c(c_\mu, A, \gamma)$. Since by assumption $s > 0$ in (2.1), for each i we have

$$\begin{aligned} (\rho(z'_i))^2 \mu(B_{\rho(z'_i)}(z'_i))^{1-2/q} &\leq c \left(\frac{\mu(B_{2R}(x_0))}{(2R)^s} \right)^{1-2/q} (\rho(z'_i))^{2+s(1-2/q)} \\ &\leq c(\mu(B_{2R}(x_0)))^{1-2/q} R^2, \end{aligned}$$

for every $2s/(2+s) < q < 2$, where $c = c(c_\mu)$. Hence

$$\begin{aligned} \int_{G(\lambda)} h^2 d\nu &\leq c\lambda^{2-q} \int_{G(\delta\lambda)} h^q d\nu + c \int_{G_{f_1}(\delta\lambda)} f_1^2 d\nu \\ (5.10) \quad &+ c(\mu(B_{2R}(x_0)))^{1-2/q} R^2 \left(\int_{G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q}. \end{aligned}$$

From now on the higher integrability result is a consequence of (5.10) and Fubini's theorem. To see this, we integrate over $G(\lambda_0)$ and use Fubini's theorem to obtain

$$\begin{aligned} \int_{G(\lambda_0)} h^{2+\varepsilon} d\nu &= \int_{G(\lambda_0)} \left(\int_{\lambda_0}^h \varepsilon \lambda^{\varepsilon-1} d\lambda + (\lambda_0)^\varepsilon \right) h^2 d\nu \\ &= \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \int_{G(\lambda)} h^2 d\nu d\lambda + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^2 d\nu, \end{aligned}$$

and now by (5.10)

$$\begin{aligned} \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \int_{G(\lambda)} h^2 d\nu d\lambda &\leq c \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon+1-q} \int_{G(\delta\lambda)} h^q d\nu d\lambda \\ &\quad + c \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \int_{G_{f_1}(\delta\lambda)} f_1^2 d\nu d\lambda \\ &\quad + c(\mu(B_{2R}(x_0)))^{1-2/q} R^2 \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \left(\int_{G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q} d\lambda. \end{aligned}$$

By Fubini's theorem again, we see that

$$\begin{aligned}
& \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon+1-q} \int_{G(\delta\lambda)} h^q d\nu d\lambda + \lambda_0^\varepsilon \int_{G(\lambda_0)} h^2 d\nu \\
&= \varepsilon \int_{G(\delta\lambda_0)} \left(\int_{\lambda_0}^{h/\delta} \lambda^{\varepsilon-1+2-q} d\lambda \right) h^q d\nu + \lambda_0^\varepsilon \int_{G(\lambda_0)} h^2 d\nu \\
&\leq \frac{\varepsilon}{\delta^{2+\varepsilon-q}(\varepsilon+2-q)} \int_{G(\lambda_0)} h^{\varepsilon+2} d\nu + \lambda_0^\varepsilon \int_{G(\delta\lambda_0)} h^2 d\nu,
\end{aligned}$$

where $c = c(A, c_\mu, \gamma)$. Observe that in the last step we also used the fact that $h^{\varepsilon+2} \leq \lambda_0^\varepsilon h^2$ in $G(\delta\lambda_0) \setminus G(\lambda_0)$. In similar fashion we obtain

$$\int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \int_{G_{f_1}(\delta\lambda)} f_1^2 d\nu d\lambda \leq \delta^{-\varepsilon} \int_{Q_{2R} \cap \Omega_T} f_1^{2+\varepsilon} d\nu,$$

and

$$\begin{aligned}
& \int_{\lambda_0}^{\infty} \varepsilon \lambda^{\varepsilon-1} \left(\int_{G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q} d\lambda \\
&\leq \left(\int_{G_{f_2}(\delta\lambda_0)} f_2^q d\mu \right)^{2/q-1} \delta^{-\varepsilon} \int_{B_{2R} \cap \Omega_T} f_2^{q+\varepsilon} d\mu \\
&\leq \delta^{-\varepsilon} \mu(B_{2R}(x_0))^{\frac{\varepsilon}{q+\varepsilon}(2/q-1)} \left(\int_{B_{2R} \cap \Omega} f_2^{q+\varepsilon} d\mu \right)^{\frac{2+\varepsilon}{q+\varepsilon}},
\end{aligned}$$

where in the final step we have used Hölder's inequality. We can now choose a positive $\varepsilon = \varepsilon(c_\mu, A, \gamma, q)$ small enough to absorb the term containing $h^{2+\varepsilon}$ into the left hand side of (5.10), and conclude that

$$\begin{aligned}
(5.11) \quad & \int_{G(\lambda_0)} h^{2+\varepsilon} d\nu \leq c(\lambda_0)^\varepsilon \int_{G(\delta\lambda_0)} h^2 d\nu + c \int_{Q_{2R} \cap \Omega_T} f_1^{2+\varepsilon} d\nu \\
& + c\mu(B_{2R}(x_0))^{(1-\frac{\varepsilon}{q+\varepsilon})(1-2/q)} R^2 \left(\int_{B_{2R} \cap \Omega} f_2^{q+\varepsilon} d\mu \right)^{\frac{2+\varepsilon}{q+\varepsilon}},
\end{aligned}$$

where $c = (c_\mu, A, \gamma)$. In case the term containing $h^{2+\varepsilon}$ is infinite, we replace h by $h_k = \min\{h, k\}$ where $k > \lambda$. Starting from (5.10) we estimate that

$$\begin{aligned}
(5.12) \quad & \int_{\{h_k > \lambda\}} h_k^{2-q} d\zeta \leq c\lambda^{2-q} \int_{\{h_k > \delta\lambda\}} d\zeta + c \int_{G_{f_1}(\delta\lambda)} f_1^2 d\nu \\
& + c(\mu(B_{2R}(x_0)))^{1-2/q} R^2 \left(\int_{G_{f_2}(\delta\lambda)} f_2^q d\mu \right)^{2/q}.
\end{aligned}$$

where $d\zeta = h^q d\nu$. Performing now as above the calculations involving Fubini's theorem yields

$$\begin{aligned} \int_{\{h_k > \lambda_0\}} h_k^{2+\varepsilon-q} d\zeta &\leq \varepsilon c \int_{\{h_k > \lambda_0\}} h_k^{2+\varepsilon-q} d\zeta + \lambda_0^\varepsilon \int_{\{h_k > \delta\lambda_0\}} h_k^{2-q} d\zeta \\ &\quad + c \int_{Q_{2R} \cap \Omega_T} f_1^{2+\varepsilon} d\nu + c\mu(B_{2R}(x_0))^{\frac{q-2}{q+\varepsilon}} R^2 \left(\int_{B_{2R} \cap \Omega} f_2^{q+\varepsilon} d\mu \right)^{\frac{2+\varepsilon}{q+\varepsilon}}. \end{aligned}$$

Now we can absorb the term containing $h_k^{2+\varepsilon-q}$ into the left hand side, and finally let $k \rightarrow \infty$ to obtain (5.11).

Finally, from the definitions of the parabolic distance and the parabolic cylinder, it is again straightforward to check that $Q_R \subset Q_{4r(z)}(z)$ for every $z \in Q_R$. Hence, by the doubling property of the measure,

$$\alpha(z) \leq \frac{\nu(Q_{2R})}{\nu(Q_R)} \frac{\nu(Q_{4r(z)}(z))}{\nu(Q_{\frac{r(z)}{5A}}(z))} \leq c_1, \quad \text{for every } z \in Q_R,$$

where $c_1 = c_1(c_\mu, A) > 0$. On the other hand, clearly $\alpha(z) \geq 1$ for every $z \in Q_{2R}$. Now (5.11) and the definition of λ_0 imply that

$$\begin{aligned} \int_{Q_R \cap \Omega_T} g^{2+\varepsilon} d\nu &\leq c_1^{\frac{2+\varepsilon}{2}} \left((\lambda_0)^\varepsilon \int_{Q_R \setminus G(\lambda_0)} h^2 d\nu + \int_{G(\lambda_0)} h^{2+\varepsilon} d\nu \right) \\ &\leq c \frac{1}{(\nu(Q_{2R}))^{\varepsilon/2}} \left(\int_{Q_{2R} \cap \Omega_T} g^2 d\nu \right)^{\frac{2+\varepsilon}{2}} + c \int_{Q_{2R} \cap \Omega_T} f_1^{2+\varepsilon} d\nu \\ &\quad + c\mu(B_{2R}(x_0))^{\frac{q-2}{q+\varepsilon}} R^2 \left(\int_{B_{2R} \cap \Omega} f_2^{q+\varepsilon} d\mu \right)^{\frac{2+\varepsilon}{q+\varepsilon}}, \end{aligned}$$

where $c = c(c_\mu, A, \gamma) > 0$. From this expression the proof can readily be completed. \square

We have now all the necessary pieces to prove global higher integrability. Note however, that because using the uniform thickness condition, needed for the reverse Hölder inequality, is valid only when close enough to the lateral boundary, the ratio between ρ_0 and the radius R of the cylinder where we want to prove higher integrability affects the constants in the final estimate.

Theorem 5.2 (Global higher integrability). *Let $u \in L^2(0, T; N^{1,2}(\Omega))$ be a parabolic quasiminimizer in Ω_T , where Ω is such that $X \setminus \Omega$ is uniformly 2-thick, and that $\eta : [0, T) \times \Omega \mapsto \mathbb{R}$, where $\eta \in W^{1,2}(0, T; N^{1,2}(\Omega))$ and $\eta(x, 0) \in N^{1,2}(\Omega)$, sets a parabolic boundary condition for u , as described in section 2.8. Let ρ_0 be the constant from Lemma 4.9.*

Suppose that we also have $\eta \in W^{1,2+\varepsilon'}(0, T; N^{1,2+\varepsilon'}(\Omega))$ for some positive ε' . Then there exists a positive constant ε and a positive constant $c = c(c_\mu, c_P, \lambda, K, \max\{1, R/\rho_0\})$, such that for every $Q_R = B_R \times \Lambda_R \subset X \times$

$(-\infty, T)$, we have

$$\begin{aligned} & \left(\frac{1}{\nu(Q_R)} \int_{Q_R \cap \Omega_T} g_u^{2+\varepsilon} d\nu \right)^{\frac{1}{2+\varepsilon}} \leq \left(\frac{c}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} g_u^2 d\nu \right)^{\frac{1}{2}} \\ & + \left(\frac{c}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} g_\eta^{2+\varepsilon} d\nu \right)^{\frac{1}{2+\varepsilon}} + \left(\frac{c}{\nu(Q_{2R})} \int_{Q_{2R} \cap \Omega_T} \left| \frac{\partial \eta}{\partial t} \right|^{2+\varepsilon} d\nu \right)^{\frac{1}{2+\varepsilon}} \\ & + \left(\frac{c}{\mu(B_{2R})} \int_{B_{2R} \cap \Omega} g_\eta^{q+\varepsilon}(x, 0) d\mu \right)^{\frac{1}{q+\varepsilon}}, \end{aligned}$$

where $\varepsilon < \varepsilon'$, $1 < q < 2$ and $q + \varepsilon < 2$.

In case Q_{2R} is such that $B_{2R} \subset \Omega$, we obtain a stronger estimate, in the sense that the second and third term on the right hand side of the above expression can be dropped, and in this case $c = c(c_\mu, c_P, \lambda, K)$. Moreover, in this case we only need to assume that u satisfies the initial condition (2.8) with some $\eta \in N^{1,2}(\Omega)$.

Proof. Assume a parabolic cylinder $Q_{2R} = B_{2R} \times \Lambda_{2R}$. In the case $B_{2R} \cap \Omega = \emptyset$, the claim is true. In the case $B_{2R} \subset \Omega$, then by Lemma 3.4, inequality (5.1) holds with $A = 2\lambda$, $f_1 = 0$ and $f_2 = g_\eta(x, 0)$ for every z' and ρ such that $Q_{2\lambda\rho}(z') \subset Q_{2R}$. Theorem 5.1 then implies the result.

Let then B_{2R} be such that $B_{2R} \cap \Omega \neq \emptyset$ and $B_{2R} \setminus \Omega \neq \emptyset$. We set $A = \max\{6\lambda^2, 2R/\rho_0\}$, where ρ_0 is the constant from Lemma 4.9. Assume z' and ρ are such that $Q_{A\rho}(z') \subset Q_{2R}$. In case $B_{2\lambda\rho}(z') \subset \Omega$, we use Lemma 3.4. In case $B_{2\lambda\rho}(z') \setminus \Omega \neq \emptyset$, since necessarily $\rho < \rho_0$, we can use Lemma 4.9. In both cases, after estimating from above on the right hand side in the former case, we obtain inequality (5.1) with $A = \max\{6\lambda^2, 2R/\rho_0\}$ and

$$f_1 = g_\eta + \left| \frac{\partial \eta}{\partial t} \right|, \quad f_2 = g_\eta(x, 0).$$

Theorem 5.1 now completes the proof. \square

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